Mathematical Tools for Parameters Identification

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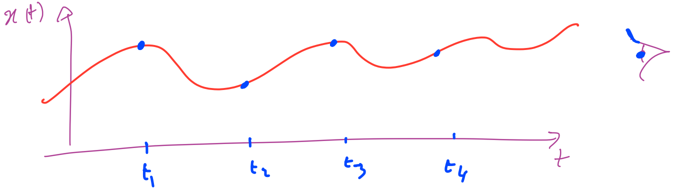

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- Hauer, J.F.; Demeure, C.J.; Scharf, L.L. (1990). "Initial results in Prony analysis of power system response signals". IEEE Transactions on Power Systems. 5: 80-89.
- Carriere, R.; Moses, R.L. (1992). "High resolution radar target modeling using a modified Prony estimator". IEEE Transactions on Antennas and Propagation. 40: 13-18.

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- $n = 1.66 + 0.25I$ => Leukemia,

Detecting Leukemia: Using Computational Electromagnetics
David Colton and Peter Monk
IEEE Computational Science Engineering archive
Volume 2 Issue 4, December 1995 Pages 46-52
IEEE Computer Society Press Los Alamos, CA, USA

Habib Ammari
An Introduction to Mathematics of Emerging Biomedical Imaging
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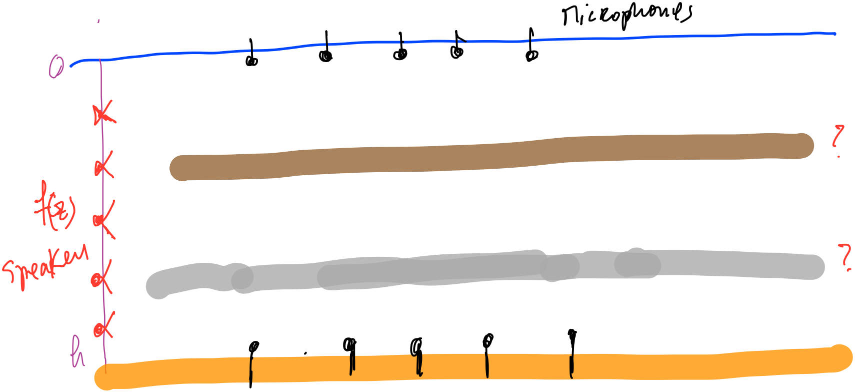

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- Fadhel Al-Musallam, A.B.,
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  SIAM J. Appl. Math. 71, (2011), 972-982.

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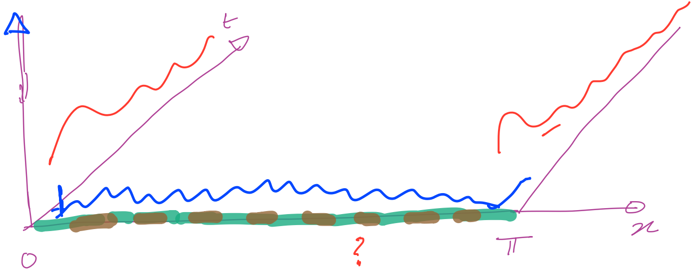

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- Recovery of the heat coefficient by two measurements, Inverse Problems and Imaging , 5, No. 4, (2011), 775-791

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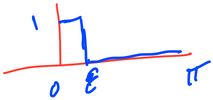
- $c_n(f)=\int_0^\pi f(x)\,\varphi(x,\lambda_n)\,dx\neq 0$ how to choose f?

$f(x) = x^\alpha$

$-\frac{1}{2} < \alpha < 0$

Large $\lambda_n \to \infty$



$\lambda_1, \dots, \lambda_N$

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- Repeat the procedure to get a_2, λ_2,....

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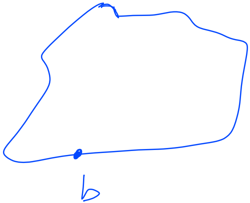
- Reconstruct $q \in C(\Omega)$

$$[D] \begin{cases} u_t\,(x,t) = \Delta u(x,t) + q(x)u(x,t) & x \in \Omega \Subset \mathbb{R}^d, \quad d \geq 2 \\ u(x,t) = 0 \qquad \text{for} \;\; x \in \partial\Omega & \\ u(x,0) = f(x) \in L^2(\Omega) & \end{cases}$$

from the map

$$f \longrightarrow \{\partial_\nu u(b,k)\}_{k \geq 1}$$



where $b \in \partial\Omega$

Borg type results on $\Delta + q$

$$\begin{cases} \Delta\varphi_n(x) + q(x)\varphi_n(x) = -\lambda_n\varphi_n(x) \\ \varphi_n(x) = 0 \quad \text{on} \qquad \partial\Omega \end{cases}$$

Result: Nachman Uhlmann, Sylvester,
$\left\{\lambda_n, \quad \frac{\partial}{\partial\nu}\varphi_n(x) \quad x \in \partial\Omega\right\}_{n\geq 1}$ determines q uniquely.

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- A.B. and Vu Kim Tuan, Inverse problems for multidimensional heat equations by measurements at a single point on the boundary, *Numer. Funct. Anal. Optim.* 30(2009), no. 11-12, 1215-1230

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6. Multiple spectrum $\frac{\partial}{\partial\nu}u(b,t) = \sum_n e^{-\lambda_n t}\sum_k c_{nk}(f)\,\frac{\partial}{\partial\nu}\varphi_{nk}(b)$

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7. $$q(x) = \frac{-\Delta\varphi_1(x) - \lambda_1\varphi_1(x)}{\varphi_1(x)} \quad \square$$

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5. for any $b \in \partial\Omega$, there is $j \geq 1$ such that $\phi_j(b) \neq 0$. Thus

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Neumann case

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7. $\Delta\phi_j(x) + q(x)\phi_j(x) = -\lambda_j \phi_j(x) \qquad \Rightarrow$ reconstruct $q(x) \qquad$ a.e.

n coefficients in the the Self-Adjoint case

$$\begin{cases} u_t(x,t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u(x,t)\right) - q(x)u(x,t), & x \in \Omega \subset \mathbb{R}^d, \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(x,0) = \psi(x) \in L^2(\Omega), \end{cases}$$

$$-q(x)\varphi_m + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} a_{ij}\right)\frac{\partial \varphi_m}{\partial x_j} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 \varphi_m}{\partial x_i \partial x_j} = -\lambda_m \varphi_m.$$

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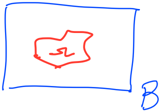

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Algebra

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   1. vanish at the origin, $\varphi_{ml}(0,\theta) = 0$ for all $m \geq 1$ for $l \in \mathbb{Z}$.

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2. ex: Laplacian on a disk. its eigenfunctions, $\varphi_{ml}(r,\theta) = J_m(rc_{ml})\, e^{il\theta}$
   1. vanish at the origin, $\varphi_{ml}(0,\theta) = 0$ for all $m \geq 1$ for $l \in \mathbb{Z}$.
   2. but $\{\varphi_{0l}(0,\theta)\}_{l \in \mathbb{Z}}$ is nonzero as predicted.

Wronskians

- Reconstruct the eigenfunctions, the N-vectors $\Phi_m(x)$, with

$$\Phi_m(x) := \left(\varphi_m(x), \frac{\partial \varphi_m}{\partial x_1}(x), \frac{\partial \varphi_m}{\partial x_n}(x), \frac{\partial^2 \varphi_m}{\partial x_1^2}(x), \cdots, \frac{\partial^2 \varphi_m}{\partial x_i \partial x_j}(x), \cdots, \frac{\partial^2 \varphi_m}{\partial x_n^2}(x \right. \tag{1}$$

proposition

Given a set of indices $k_1 < k_2 < \cdots < k_N$, such that λ_{k_j}, $j = 1, \cdots, N$, are distinct and nonzero. Then the set

$$E := \left\{ x \in \Omega : \det \left[\Phi_{k_1}^T(x),\ \Phi_{k_2}^T(x),\ \ldots,\ \Phi_{k_N}^T(x) \right] \neq 0 \right\} \tag{2}$$

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- The main idea here is to use the constant rank theorem

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We can reconstruct the shape of Ω, the coefficients $a_{i,j}$ and q from measurements $\psi_l \longrightarrow \{u(b,k)\}_{k,l\in\mathbb{N}}$ at any single point $b \in \Omega$.

A.B. ; Tuan, Vu Kim; Hoang, Nguyen
The recovery of a parabolic equation from measurements at a single point.
Evol. Equ. Control Theory 7 (2018), no. 2, 197-216.

Divergence type

- Consider the parabolic equation

$$\begin{cases} \partial_t u(x,t) = \nabla.\left(p(x)\nabla u(x,t)\right), & t>0, \quad x\in\Omega, \\ u(x,t)=0, & x\in\partial\Omega, \\ u(x,0)=a(x), \end{cases} \tag{3}$$

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- from a sequence of observations of the solution $\{u(b,k)\}_{k\in\mathbb{N}}$ taken at an arbitrary single point $b \in \Omega$.

Method

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We can reconstruct the shape of Ω and the diffusion coefficient p from measurements $\left\{\frac{\partial}{\partial\nu}u^{\psi_l}(b,k)\right\}_{k,l\in\mathbb{N}}$ *at any single point $b \in \Omega$.*

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- A.B and Tuan, Vu Kim One point recovery of a parabolic equation. J. Math. Anal. Appl. 463 (2018), no. 1, 161-166.

Distillation membrane

$$
\begin{cases}
\partial_t f(t,x,y) - \alpha_f \Delta f(t,x,y) + \beta_f \partial_y f(t,x,y) = 0 & t>0,\ (x,y)\in\Omega_1,\\
\partial_t p(t,x,y) - \alpha_p \Delta p(t,x,y) + \beta_p \partial_y p(t,x,y) = 0 & t>0,\ (x,y)\in\Omega_2,\\
\partial_x f(t,\ell_1,y) = \gamma_f \left(f(t,\ell_1,y) - p(t,\ell_1,y)\right) & t>0,\ y\in(0,\ell_2),\\
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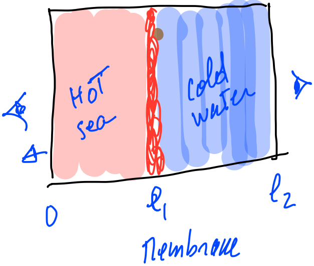
\partial_x f(t,0,y) = 0 & t>0,\ y\in(0,\ell_2),\\
\partial_x p(t,\delta_m+2\ell_1,y) = 0 & t>0,\ y\in(0,\ell_2),\\
p(t,x,0) = T_p(t,x) & t>0,\ \ x\in(\delta_m+\ell_1,\delta_m+2\\
\partial_y p(t,x,\ell_2) = 0 & t>0,\ \ x\in(\delta_m+\ell_1,\delta_m+2\\
f(t,x,0) = T_f(t,x) & t>0,\ x\in(0,\ell_1),\\
\partial_y f(t,x,\ell_2) = 0 & t>0,\ x\in(0,\ell_1),\\
f(0,x,y) = f_0(x,y) & (x,y)\in\Omega_1,\\
p(0,x,y) = p_0(x,y) & (x,y)\in\Omega_2,
\end{cases}
\qquad (4)
$$



Recover the temperature on both sides of the membrane, that is $f(t,\ell_1,y)$ and $p(t,\ell_1+\delta_m,y)$, by external boundary measurements.

Thank you