Recovery of the fractional diffusion equation from a single boundary measurement

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Consider a fractional diffusion process on a finite length rod

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u(x,t) = u_{xx}(x,t) - q(x)u(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = a(t), \\ u(x,0) = 0. \end{cases} \tag{1}$$

Here ${}_0^C\mathcal{D}_t^\alpha u(x,t)$, $0 < \alpha < 1$, denotes the Caputo derivative

$${}_0^C\mathcal{D}_t^\alpha u(x,t) = \int_0^t \frac{(t-\xi)^{-\alpha}}{\Gamma(1-\alpha)} \frac{\partial}{\partial \xi} u(x,\xi)\, d\xi.$$

Fractional diffusion equations appear in probability, random walk, and finance, as they help describe diffusion processes in presence of long range interactions.

Inverse Problem

We are concerned with the recovery of the diffusion coefficient $q(x) \in L_1(0,1)$ from the measurement of $u_x(1,t) = b(t)$, given $u(0,t) = a(t)$, $t \in (0,\infty)$, at one end of the rod only.

This problem is close in spirit with the boundary control (BC) method, where $a(t)$ is a given input on the boundary and we can observe its response $b(t)$ also on the same boundary. However the BC method is usually applied to wave equations because of their finite wave speed propagation.

A. Ramm (2001)

A similar problem for a heat process (when ${}_0^C\mathcal{D}_t^\alpha u(x,t)$ is replaced by $u_t(x,t)$) has been considered by Ramm, where $a(t)$ was assumed to be a pulse-type function

$$a(t) = 0 \quad \text{for} \quad t > T, \quad \int_0^T a(t)\,dt < \infty. \quad a(t) \not\equiv 0, \tag{2}$$

By using the Laplace transform

$$F(s) = (\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t)\,dt, \tag{3}$$

he could recast the heat equation into a Sturm-Liouville problem

$$\begin{cases} sU(x,s) = U''(x,s) - q(x)U(x,s), & 0 < x < 1, \\ U(0,s) = 0, \quad U(1,s) = A(s), & \\ U(x,0) = 0, & \end{cases} \tag{4}$$

A. Ramm (2001) continued

and $U^{'}(1,s) = B(s)$, where $U(x,s), A(s)$, and $B(s)$ were the Laplace transforms of $u(x,t), a(t)$, and $b(t)$, respectively. It follows then the Dirichlet eigenvalues of the problem (4) should be the zeros of $A(s)$. However, if $a(t) > 0$ on $(0,T)$, then $A(s) \neq 0$ for any real s. Thus without the eigenvalues of the associated Sturm-Liouville problem, it is not clear how one would pursue his method to recover $q(x)$.

Let $\varphi(x,\lambda)$ solve the following initial value problem

$$\begin{cases} \varphi''(x,\lambda) - q(x)\varphi(x,\lambda) = -\lambda\varphi(x,\lambda), & 0 < x < 1, \\ \varphi(0,\lambda) = 0, \quad \varphi'(0,\lambda) = 1. \end{cases} \tag{5}$$

Then $\varphi(1,\lambda)$ is an entire function of exponential type, and its zeros $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ are the eigenvalues of the Sturm-Liouville boundary value problem

$$\begin{cases} \varphi''(x,\lambda_n) - q(x)\varphi(x,\lambda_n) = -\lambda_n\varphi(x,\lambda_n), & 0 < x < 1, \\ \varphi(0,\lambda_n) = 0, \quad \varphi(1,\lambda_n) = 0. \end{cases} \tag{6}$$

For simplicity we denote $\varphi_n(x) := \varphi(x,\lambda_n)$, the n-th eigenfunction associated with the eigenvalue λ_n.

Gelfand-Levitan Theory

Recall that to determine a potential q uniquely we need only the complete set of eigenvalues and norming constants $\{\lambda_n, \ \|\varphi_n\|_2\}_{n>0}$, in order to build the spectral function, a celebrated result of the Gelfand-Levitan theory. Define the kernel

$$F(x,t) = \sum_{n=1}^{\infty} \left(\frac{\sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_n}t)}{\lambda_n \|\varphi_n\|_2^2} - 2\sin(n\pi x)\sin(n\pi t) \right),$$

and solve the Fredholm integral equation, for each fixed $0 < x < 1$,

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \qquad 0 \le t \le x \le 1,$$

to obtain $K(x,t)$, which is known to be absolutely continuous and the function $q(x)$ is then given by $q(x) = 2\frac{d}{dx}K(x,x)$ a.e. Hence, the knowledge of the complete spectral data $\{\lambda_n, \ \|\varphi_n\|_2\}_{n>0}$ determines $q(x)$ uniquely.

Boundary Spectral Data

We show that the full boundary spectral data $\left\{\lambda_n, \varphi_n'(1)\right\}_{n>0}$ allow us to recover all the norming constants $\|\varphi_n\|_2$. The eigenvalues λ_n are the zeros of $\varphi\left(1,\lambda\right)$, and the boundary function $\varphi\left(1,\lambda\right)$, as an entire function of exponential type of order $\frac{1}{2}$, is determined uniquely, up to a multiplicative constant, by its zeros $\{\lambda_n\}_{n>0}$

$$\varphi\left(1,\lambda\right) = C\prod_{n>0}\left(\frac{\lambda_n-\lambda}{\pi^2 n^2}\right). \tag{7}$$

The boundary conditions $\varphi(0,\lambda)=0$, $\varphi_x(0,\lambda)=1$ guarantee that $C=1$ in (7), so $\varphi(1,\lambda)$ is determined uniquely by its zeros $\{\lambda_n\}_{n>0}$.

Boundary Spectral Data (continued)

The norms $\|\varphi_n\|_2$ can be determined from $\varphi(1, \lambda)$ in (7) and $\varphi_n^{'}(1)$ by

$$\|\varphi_n\|_2^2 = \varphi_n'(1)\varphi_\lambda(1, \lambda_n). \tag{8}$$

Homogeneous Boundary Condition

If we set

$$v(x,t) = u(x,t) - xa(t), \tag{9}$$

then $v(x,t)$ solves a fractional diffusion equation with a source but with the homogenous boundary and initial conditions

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha v\,(x,t) = v_{xx}(x,t) - q(x)v(x,t) - x\ {}_0^C\mathcal{D}_t^\alpha a(t) - xq(x)a(t), \\ v(0,t) = 0, \quad v(1,t) = 0, \\ v(x,0) = 0, \end{cases} \tag{10}$$

while $v_x(1,t) = b(t) - a(t)$.

Separation of Variables

The general solution $v(x,t)$ of (10) can be obtained by separation of variables

$$v(x,t) = \sum_{n>0} c_n(t)\varphi_n(x), \tag{11}$$

where $c_n(t)$ satisfies the initial-value problem

$${}_0^C\mathcal{D}_t^\alpha c_n(t) = -\lambda_n c_n(t) - p_n\, {}_0^C\mathcal{D}_t^\alpha a(t) - q_n a(t), \quad c_n(0) = 0, \tag{12}$$

with

$$p_n = \frac{\int_0^1 x\varphi_n(x)\,dx}{\|\varphi_n\|_2^2}, \quad q_n = \frac{\int_0^1 xq(x)\varphi_n(x)\,dx}{\|\varphi_n\|_2^2}. \tag{13}$$

Series Representation of Solution

Applying the Laplace transform to (12) and recalling that

$$\mathcal{L}\left({}_{0}^{C}\mathcal{D}_{t}^{\alpha}f(t)\right)(s)=s^{\alpha}F(s)-s^{\alpha-1}f(0), \tag{14}$$

with $c_n(0)=a(0)=0$ we obtain

$$C_n(s)=-\frac{p_n s^{\alpha}+q_n}{s^{\alpha}+\lambda_n}A(s)=-p_nA(s)+(\lambda_n p_n-q_n)\frac{1}{s^{\alpha}+\lambda_n}A(s). \tag{15}$$

$$\mathcal{L}\left(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\right)(s)=\frac{1}{s^{\alpha}+\lambda}, \tag{16}$$

where $E_{\alpha,\beta}(x)$ is the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+\beta)},$$

we get

$$c_n(t)=-p_na(t)+(\lambda_np_n-q_n)\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})a(\tau)\,d\tau.$$

Series Representation of Observation

$$u(x,t) = \sum_{n>0} (\lambda_n p_n - q_n)\varphi_n(x) \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha)\, a(\tau)\, d\tau. \tag{17}$$

Consequently,

$$b(t) = \sum_{n>0} (\lambda_n p_n - q_n)\varphi_n'(1) \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) a(\tau)\, d\tau. \tag{18}$$

$$\|\varphi_n\|_2^2 \,(q_n - \lambda_n p_n) = \varphi_n'(1). \tag{19}$$

Thus, (18) becomes

$$b(t) = -\sum_{n>0} \frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha)\, a(\tau)\, d\tau. \tag{20}$$

Convergence

$$E_{\alpha,\alpha}(-\lambda_n t^\alpha) = O(n^{-4}),$$

uniformly in t on any compact subinterval of $(0, \infty)$. On the other hand,

$$\lambda_n \sim (n\pi)^2, \quad \left[\varphi_n'(1)\right]^2 \sim 1, \quad \|\varphi_n\|_2^{-2} \sim 2\pi^2 n^2. \qquad (21)$$

Consequently, the series (20) converge for any $t > 0$.

Data Extraction

Representation (20) is the key to our inverse problem. First, since $E_{\alpha,\alpha}(-t)$ is positive on $(0,\infty)$, formula (20) says that if $a(t)$ is positive on $(0,\infty)$, then the observation $b(t)$ must be negative on $(0,\infty)$. Next, since $\varphi_n'(1) \neq 0$ for any n, the series (20) contains all eigenvalues λ_n. We now explain how to extract the eigenvalues λ_n and the coefficients $\frac{(\varphi_n'(1))^2}{\|\varphi_n\|_2^2}$ from the observation $b(t)$. If we could apply the Laplace transform to (20) we would get

$$B(s) = -\sum_{n>0}^{\infty} \frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2} \frac{1}{s^\alpha + \lambda_n} A(s). \tag{22}$$

Formula (22) would say that $-\lambda_n$ are exactly the poles of $-\frac{B(s^{1/\alpha})}{A(s^{1/\alpha})}$ with the corresponding residues $\frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2}$. Thus, $\frac{B(s^{1/\alpha})}{A(s^{1/\alpha})}$ would reveal all eigenvalues λ_n and spectral boundary values $\frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2}$.

Unfortunately, from the asymptotic formula (21) we see that

$$\lim_{n\to\infty} \frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2} \frac{1}{s+\lambda_n} = 2,$$

hence the series (22) is in fact divergent, which means that the Laplace transform of $b(t)$ does not exist.

Modified Function $d(t)$

To overcome this difficulty we construct a function $d(t)$, similar to $-b(t)$, but with 2α instead of α and $\lambda_n, \varphi_n'(1)$, and $\|\varphi_n\|_2$ being replaced by their asymptotics (21)

$$d(t) = \sum_{n>0}^{\infty} 2n^2\pi^2 t^{2\alpha-1} E_{2\alpha,2\alpha}(-n^2\pi^2 t^{2\alpha}) * a(t). \qquad (23)$$

As done to (20), the series (23) is shown to converge for all $t > 0$.

The Laplace transform yields

$$\frac{s^{1/\alpha}\mathcal{L}(b(t)+d(t))(s^{1/\alpha})}{A(s^{1/\alpha})}=\sum_{n>0}^{\infty}\left(\frac{2n^2\pi^2}{s^2+n^2\pi^2}-\frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2}\frac{1}{s+\lambda_n}\right). \quad (24)$$

Thus, $s^{1/\alpha}\frac{\mathcal{L}(b(t)+d(t))(s^{1/\alpha})}{A(s^{1/\alpha})}$ has real poles at $-\lambda_n$ with residues $-\frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2}$, and complex poles at $in\pi$, and so real poles of (24) yield all necessary boundary spectral data $\left\{\lambda_n, \frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2}\right\}_{n>0}$.

But

$$\|\varphi_n\|_2^2 = \frac{[\varphi_n'(1)]^2}{\|\varphi_n\|_2^2}[\varphi_\lambda(1,\lambda_n)]^2.$$

Since $\varphi(1,\lambda)$ is known, then $\varphi_\lambda(1,\lambda_n)$ is also known and therefore, all $\|\varphi_n\|_2$ can be determined.

Theorem

Let the boundary condition $u(1,t) = a(t)$ be a nontrivial nonnegative and bounded function with $a(0) = 0$. Then the single measurement of $u_x(1,t) = b(t)$, $t \in (0,\infty)$, determines $q(x) \in L_1(0,1)$ uniquely.

Consider now the diffusion equation with the Neumann boundary conditions

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u\,(x,t) = u_{xx}(x,t) - q(x)u(x,t),\ 0<x<1,\ t>0, 0<\alpha<1, \\ u_x(0,t)=0, \quad u_x(1,t)=b(t), \\ u(x,0)=0. \end{cases} \tag{25}$$

The inverse problem is to recover the diffusion coefficient $q(x) \in L_1(0,1)$ from the reading of $u(1,t)=a(t),\ 0<t<\infty$, at one end of the rod only.

Uniqueness

Theorem

Let $u_x(1,t) = b(t)$ be a nontrivial nonnegative and bounded function with $b(0) = 0$. Then a single measurement of $u(1,t) = a(t)$ on $(0,\infty)$ determines $q(x) \in L_1(0,1)$ uniquely.

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u(x,t) = u_{xx}(x,t) - q(x)u(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = f(x). \end{cases} \tag{26}$$

Initial-to-Boundary Inverse Diffusion Problem: Choose finitely many special initial distributions $u(x,0) = f(x)$ so that the boundary readings $u_x(0,t), u_x(1,t)$ recover the diffusion coefficient $q(x)$ uniquely.

Eigenfunctions Expansion

Any $f \in L_2(0,1)$ can be expanded uniquely into the generalized Fourier series

$$f(x) = \sum_{n>0} c_n \varphi_n(x), \text{ where } c_n = \frac{(f,\ \varphi_n)}{\|\varphi_n\|_2^2}. \tag{27}$$

Separation of variables method yields the general solution $u^f = u$ of (26) in the series form

$$u(x,t) = \sum_{n>0} c_n E_\alpha\left(-\lambda_n t^\alpha\right) \varphi_n(x), \tag{28}$$

and its spatial derivative in the form

$$u_x(x,t) = \sum_{n>0} c_n E_\alpha\left(-\lambda_n t^\alpha\right) \varphi_n'(x). \tag{29}$$

Uniform convergence

The asymptotic formulas

$$c_n = O(n^2), \quad \varphi_n(x) = O\left(\frac{1}{n}\right), \quad \varphi_n'(x) = O(1), \quad x \in [0,1], \tag{30}$$

$$E_\alpha\left(-\lambda_n t^\alpha\right) = O\left(\frac{1}{n^4}\right), \quad t > 0, \tag{31}$$

yield the uniform convergence of the series (28) and (29) in $x \in [0,1]$, for each $t > 0$.

Boundary readings

The uniform convergence of the series (28) and (29) on $[0, 1]$ allows us to represent the readings at the boundary points $x = 0$ and $x = 1$ as series of Mittag-Leffler functions

$$u_x(0,t) = \sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)$$

$$u_x(1,t) = \sum_{n\geq 0} c_n E_\alpha\left(-\lambda_n t^\alpha\right)\varphi_n'(1)$$

Analyticity of the boundary measurements

Lemma

The boundary readings are restrictions on $(0, \infty)$ *of functions, analytic in the angle* $0 \le |\arg(t)| < \min\left\{\pi, \frac{\pi}{\alpha} - \frac{\pi}{2}\right\}$.

Lemma says that reading $u_x(0, t)$ on some time interval $t \in (T_0, T_1)$ will determine uniquely the boundary values $u_x(0, t)$ for any $t > 0$ by analytic continuation. Similarly for $u_x(1, t)$.

Extracting boundary spectral data

Problem: How to get boundary spectral data $\{\lambda_n, \varphi_n'(1)\}_{n>0}$ **from boundary measurements.**

$$s^{(1-\alpha)/\alpha}\mathcal{L}(u_x(0,t))(s^{1/\alpha}) = \sum_{n>0} \frac{c_n}{s+\lambda_n},$$

$$s^{(1-\alpha)/\alpha}\mathcal{L}(u_x(1,t))(s^{1/\alpha}) = \sum_{n>0} \frac{c_n\varphi_n'(1)}{s+\lambda_n}.$$

Thus, from $\mathcal{L}(u_x(0,t))(s)$ and $\mathcal{L}(u_x(1,t))(s)$, and therefore, from $u_x(0,t)$ and $u_x(1,t)$, $t \in (T_0, T_1)$, we can extract λ_n and $\varphi_n'(1)$ whenever $c_n \neq 0$.

Main question: Completeness of data

We can extract **boundary spectral data** $\{\lambda_n, \varphi_n'(1)\}_{c_n \neq 0}$ from lateral measurements by the Laplace transform.

How to choose one (or few) initial distributions f such that $c_n^f = \frac{(f, \varphi_n)}{\|\varphi_n\|_2^2} > 0$ for **all** $n > 0$ and for at least one f?

Proposition. Let $q \in L_1(0,1)$ and $f(x) = x^\alpha$, where $\alpha \in (-1/2, 0)$, then there is $N > 0$, such that $c_n \neq 0$ for $n \geq N$.
Proof.

$$\varphi_n(x) = \frac{\sin \pi n x}{\pi n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

$$\int_0^1 x^\alpha \varphi_n(x)\, dx = \frac{c(\alpha)}{n^{\alpha+2}} + o\left(\frac{1}{n^{\alpha+2}}\right),$$

$$c(\alpha) := \frac{1}{\pi^{\alpha+2}} \int_0^\infty \xi^\alpha \sin \xi d\xi > 0 \qquad (32)$$

No large eigenvalues are missing

$$\|\varphi_n\|_2^2 \, c_n \, n^{\alpha+2} \to c(\alpha) \neq 0, \quad \text{as} \quad n \to \infty.$$

Thus the choice $f(x) = x^\alpha$, $-1/2 < \alpha < 0$, guarantees **the existence of** N **such that** $c_n \neq 0$ **for** $n \geq N$.

If no extra condition is imposed on q then the actual value N cannot be estimated.

Counting missing eigenvalues

The asymptotics of eigenvalues

$$\sqrt{\lambda_n} = \pi n + \frac{a_1}{n} + o\left(\frac{1}{n}\right), \qquad a_1 = \frac{1}{2}\int_0^1 q(x)dx \ .$$

Let M be the number starting from which all recovered eigenvalues larger than $\pi^2(M-1/2)^2$ fall in enclosures of the form $(\pi^2(k-1/2)^2,\ \pi^2(k+1/2)^2)$, where $k \geq M$. More importantly each of these intervals contains exactly one recovered eigenvalue. The existence of such a value M comes from the asymptotics of the eigenvalues and the Proposition which says that after a certain N no eigenvalue is missing.
Now suppose there are only L eigenvalues recovered in $(-\infty, \pi^2(M-1/2)^2]$. Then only $M-L$ eigenvalues are missing and so by knowing M, we readily know the number of missing eigenvalues.

Example

For example, starting for $k \geq 100$, each of the intervals $(\pi^2(k-1/2)^2, \pi^2(k+1/2)^2)$ contains exactly one recovered eigenvalue, which can be ranked tentatively as λ_k, and none falls outside. If by counting, we have found only 80 eigenvalues less than $\pi^2(100-1/2)^2$, it means that we are missing exactly 20 eigenvalues. The difficulty in ranking here is that the missing eigenvalues are not necessarily below λ_{100}. That is why it is important to first recover all the missing eigenvalues and only after that can the ranking be finalized.

Extracting missing eigenvalues

We start off with the following observation.

Let $x_{0,n}$ be the first positive zero of $\varphi_n(x)$. Then all $\varphi_k(x)$, $k = 1, 2, \cdots, n$, do not change sign on $(0, x_{0,n})$, and therefore, if we choose

$$f(x) = \chi_{(0,\nu)}(x) = \begin{cases} 1, & 0 < x < \nu, \\ 0, & \nu \le x, \end{cases} \quad , 0 < \nu < x_{0,n},$$

then coefficients $c_k^{\chi_{(0,\nu)}} = \|\varphi_k\|_2^{-2} \int_0^\nu \varphi_k(x)dx \neq 0$ for $k = 1, ..., n$.

Moreover, $n \to \infty$ if $\nu \to 0+$.

Stopping rule

We start by reading $u_x^{\chi_{(0,\nu_1)}}(0,t)$, $0 < \nu_1 < 1$, to find missing eigenvalues. If after that eigenvalues are still missing, we go to the next $u_x^{\chi_{(0,\nu_2)}}(0,t)$, $\nu_2 < \frac{\nu_1}{2}$, and so on $\nu_k < \frac{\nu_{k-1}}{2}$ until we have recovered all the $M - L$ missing eigenvalues. This process stops at some point and so is finite. After that, we can finally rearrange and rank the eigenvalues in increasing order.

Estimate N when $\|q\|_1$ is known

Assume the L_1-norm of q is known a priori to be bounded by Q, the actual value N then can be found a priori

Proposition. Let $\|q\|_1 \leq Q$ and $f(x) = x^\alpha$, where $\alpha \in (-1/2, 0)$. Then $c_n \neq 0$ for $\lambda_n \geq N$, where

$$N = \max\left\{4Q^2, \frac{(c(\alpha))^{\frac{2}{\alpha}}}{2^{\frac{3}{\alpha}}\pi^{2-\frac{1}{\alpha}}}\left[1 + \frac{\pi}{\alpha+1}Q\right]^{\frac{-2}{\alpha}}\right\}, \qquad (33)$$

$$c(\alpha) = \frac{1}{\pi^{\alpha+2}}\Gamma(\alpha+1)\sin\left((\alpha+1)\frac{\pi}{2}\right).$$

Estimate $\|q\|_1$ when $q \geq 0$

Observe if $q \geq 0$ then $||q||_1 = \int_0^1 q(x)dx$, a constant that appears in the asymptotics of the eigenvalues

$$\begin{aligned} \sqrt{\lambda_n} &= \pi n + \frac{a_1}{n} + o\left(\frac{1}{n}\right) \\ a_1 &= \frac{1}{2}\int_0^1 q(x)dx \ . \end{aligned}$$

Thus obviously if it is a priori known that $q \geq 0$ then $||q||_1$ can be computed from the recovered eigenvalues.

$$\|q\|_1 = 2 \lim_{n\to\infty} \left(\sqrt{\lambda_n} - \pi n\right) n.$$

Estimate $\|q\|_1$ when $q(x) \geq -C$

This case can also be reduced to the previous case by considering translated problem

$$-y''(x) + \widetilde{q}(x)y(x) = \widetilde{\lambda}y(x),$$

where $\widetilde{q} := q + C \geq 0$, $\widetilde{\lambda} := \lambda + C$ and the same boundary conditions. Thus we would recover $\widetilde{q}$ first and then q by subtracting C. Since $\|q(x) + C\|_1 = 2\lim_{n\to\infty}\left(\sqrt{C + \lambda_n} - \pi n\right) n$, and $\|q(x)\|_1 \leq \|q(x) + C\|_1 + \|C\|_1 = \|q(x) + C\|_1 + C$, we have the following

$$\|q(x)\|_1 \leq Q := C + 2\lim_{n\to\infty}\left(\sqrt{C + \lambda_n} - \pi n\right) n.$$

If $q(x) \geq -C$ then two measurements are sufficient!

If $q(x) > -C$ then the Mittag-Leffler series of $u_x^{x^\alpha}(0,t)$ contains all $\lambda_n \geq N$, where N is defined by (33), and Q is defined through C. Denote

$$\nu = \frac{\pi}{\sqrt{N + C}}. \tag{34}$$

Then $u_x^{\chi_{(0,\nu)}}(0,t)$ carries all $\lambda_n \leq N$ and two initial conditions x^α and $\chi_{(0,\nu)}(x)$ are enough to recover the full boundary spectral data, and therefore, the diffusion coefficient q.

Uniqueness

Theorem

Consider the fractional diffusion equation (26) where $q \in L_1(0,1)$. *Then* q *can be recovered uniquely from a finite number of readings at* $x = 0$ *and* $x = 1$ *for* $0 \leq T_0 < t < T_1 \leq \infty$. *If a lower bound* $q(x) > -C$ *is known a priori, then at most two measurements are enough to recover* q *uniquely.*

Neumann case

Similar results have been obtained for Neumann boundary conditions (and Robin boundary conditions) for one-dimensional fractional diffusion equations.

Similar results have been obtained for Neumann boundary conditions (and Robin boundary conditions) for one-dimensional fractional diffusion equations.

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THANK YOU